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Anomaly Matching Conditions in Supersymmetric Gauge Theories

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Abstract

Sufficient conditions are proven for 't Hooft's consistency conditions to hold at points in the moduli space of supersymmetric gauge theories. Known results for anomaly matching in supersymmetric QCD are rederived as a sample application of the results. The results can be used to show that the anomaly matching conditions hold for s-confining theories.

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One important constraint on the moduli space of vacua of supersymmetric gauge theories [1] is that the massless fermions in the low energy theory should have the same flavor anomalies as the fundamental fields, i.e. the 't Hooft consistency conditions should be satisfied [2]. The computation of the flavor anomalies at a point in the moduli space can often be quite complicated. In this paper we derive some general conditions which guarantee that 't Hooft's consistency conditions are satisfied. The results are applied to supersymmetric QCD, and agree with known results for this case. They can also be used to show that 't Hooft consistency conditions are satisfied for s-confining theories [3]. Other applications will be discussed in a longer publication [4]

The analysis makes use of the result that the classical moduli space $\mathcal{M}_{\rm cl}$ of a supersymmetric gauge theory is the algebraic quotient $U /\!\!/ G$ of the space U of critical points $W' = 0$ of the superpotential W of the fundamental theory by G, the complexification of the gauge group G_r [4,5]. This follows from the fact that a supersymmetric vacuum state is contained in a closed G-orbit, a result proved in Ref. [5]. The relation of supersymmetric vacua to closed orbits, and a more detailed discussion of the construction of $\mathcal{M}_{\rm cl}$ from an algebraic geometry viewpoint will be given in [4]. There are several subtleties in the construction not discussed in [5] which are relevant for anomaly matching.

The fundamental theory, such as supersymmetric QCD, will be referred to as the ultraviolet (UV) theory. The massless degrees of freedom that characterize the moduli space $\mathcal{M}$ will be referred to as the infrared (IR) theory. We will use $\phi \in U$ to represent a constant field configuration in the UV theory. Supersymmetric vacua are characterized by the values of $\hat\phi \in V$, where V is a vector space spanned by gauge invariant polynomials $\hat\phi^i(\phi)$ constructed out of the fundamental fields ϕ^i. The classical moduli space $\mathcal{M}_{\rm cl}$ is an algebraic set in V (see, e.g. [5]). There is a natural map $\pi : U \to \mathcal{M}_{\rm cl}$. The G-orbit of a point $\phi \in U$ will be denoted by $G\phi$. The tangent space at a point p in X will be denoted by T_pX, so that $T_\phi U$ is the tangent space at ϕ in the UV theory, and $T_{\hat\phi}\mathcal{M}$ is the tangent space at $\hat\phi$ in the IR theory. The differential of π at ϕ, π'_ϕ, gives a map from $T_\phi U \to T_{\hat\phi}\mathcal{M}_{\rm cl}$. In the following discussion, all terms (such as closed, open, dimension, etc.) are in the algebraic geometry sense. We

shall also assume that the complexified gauge group is a reductive group. For example in supersymmetric QCD with $N_F = N_C$ and no tree-level superpotential, the UV fields are the quarks $Q^{i\alpha}$ and the antiquarks $\tilde{Q}_{j\beta}$ that span a vector space U of dimension $2N_c^2$. The IR fields are the mesons M^i_j, a baryon B and antibaryon $\tilde{B}$, that span the vector space V of dimension $N_c^2 + 2$. The classical moduli space $\mathcal{M}_{\rm cl}$ is the algebraic set $\det M = B\tilde{B}$ contained in V. The map π takes $\phi = (Q^{i\alpha},\ \tilde{Q}_{j\beta})$ to $\hat{\phi} = (M^i_j,\ B,\ \tilde{B})$, where

$$M^i_j = Q^{i\alpha}\tilde{Q}_{j\alpha}, \qquad B = \det Q, \qquad \tilde{B} = \det \tilde{Q}. \tag{1}$$

The anomaly matching theorem given below requires that the map $\pi'_\phi : T_\phi U \to T_{\hat{\phi}}\mathcal{M}_{\rm cl}$ be surjective, and that $\ker \pi'_\phi = T_\phi\,(G\phi)$. Note that since π is gauge invariant, one always has $T_\phi\,(G\phi) \subseteq ker\pi'_\phi$. The following result proved in Ref. [4] establishes sufficient conditions for these requirements to hold.

Theorem I: Assume that G is totally broken at ϕ_0, i.e. $\mathrm{Lie}\,(G)\,\phi_0 \cong \mathrm{Lie}\,(G)$, and that $G\phi_0$ is closed in U. Then $\ker \pi'_{\phi_0} = T_{\phi_0}\,(G\phi_0) = \mathrm{Lie}\,(G)$, π'_{ϕ_0} is onto, and $\pi(\phi_0)$ is a smooth point of $\mathcal{M}_{\rm cl}$.

Note that the condition that $G\phi_0$ be closed is equivalent to the statement that this orbit contains a point satisfying the D-flatness conditions in Wess-Zumino gauge.

The anomaly matching theorem is

Theorem II: Let $\mathcal{M}_{\rm cl}$ be the classical moduli space of a supersymmetric gauge theory with gauge group G and flavor symmetry F. It is assumed that the gauge theory has no gauge or gravitational anomalies, and the flavor symmetries have no gauge anomalies. Let $\hat{\phi}_0 \in \mathcal{M}_{\rm cl}$ be a point in the classical moduli space. Assume there is a point $\phi_0 \in U$ in the fiber $\pi^{-1}(\pi(\phi_0))$ of $\hat{\phi}_0$ such that

(a) G is completely broken at ϕ_0, so that $\mathrm{Lie}\,(G)\,\phi_0 \cong \mathrm{Lie}\,(G)$.

(b) $\ker \pi'_{\phi_0} = \mathrm{Lie}\,(G)\,\phi_0$ and π'_{ϕ_0} is surjective.

If a subgroup $H \subseteq F$ is unbroken at $\hat{\phi}_0$, then the 't Hooft consistency conditions for the F^3 flavor anomalies and the F gravitational anomalies are satisfied.

For the purposes of the proof, it is convenient to write the original flavor symmetry as $F' \times R$, where R is the R-symmetry, and F' now contains only non-R symmetries. We first prove anomaly matching when $H \subseteq F'$, and then prove consistency for anomalies that include the R symmetry. (Note that the unbroken R symmetry might be a linear combination of the original R symmetry and some generator in F'.)

Since H is unbroken at $\hat{\phi}_0$, $\hat{\phi}_0$ is H-invariant

$$\mathrm{Lie}\,(H)\,\hat{\phi}_0 = 0. \tag{2}$$

The map $\pi : U \to \mathcal{M}_{\mathrm{cl}}$ commutes with the flavor symmetries, so

$$0 = \mathrm{Lie}\,(H)\,\hat{\phi}_0 = \mathrm{Lie}\,(H)\,(\pi\,(\phi_0)) = \pi'_{\phi_0}\,(\mathrm{Lie}\,(H)\,\phi_0)\,. \tag{3}$$

Thus, by (a) and (b)

$$\mathrm{Lie}\,(H)\,\phi_0 \subseteq \ker \pi'_{\phi_0} = \mathrm{Lie}\,(G)\,\phi_0. \tag{4}$$

This implies that given any $\mathfrak{h} \in \mathrm{Lie}\,(H)$, there is a unique $\mathfrak{g}\,(\mathfrak{h}) \in \mathrm{Lie}\,(G)$ such that

$$\mathfrak{h}\phi_0 = -\mathfrak{g}\,(\mathfrak{h})\phi_0, \tag{5}$$

where the minus sign is chosen for convenience. It is straightforward to check that the map $\mathrm{Lie}\,(H) \to \mathrm{Lie}\,(G)$ given by $\mathfrak{h} \to \mathfrak{g}\,(\mathfrak{h})$ is a Lie-algebra homomorphism,

$$\mathfrak{g}\,([\mathfrak{h}_1, \mathfrak{h}_2]) = [\mathfrak{g}\,(\mathfrak{h}_1)\,, \mathfrak{g}\,(\mathfrak{h}_2)]\,. \tag{6}$$

This allows us to define a new "star" representation of $\mathrm{Lie}\,(H)$ in U

$$\mathfrak{h}^* \equiv \mathfrak{h} + \mathfrak{g}\,(\mathfrak{h}). \tag{7}$$

Since $\mathrm{Lie}\,(G)\,\phi_0 \subseteq \ker \pi'_{\phi_0}$, the new $\mathrm{Lie}\,(H)$ representation on $T_{\hat{\phi}_0}\mathcal{M}_{\mathrm{cl}}$ defined by $\pi'_{\phi_0}\mathfrak{h}^*$ agrees with the original one. Thus the $\mathfrak{h}^*$-anomalies computed at $\hat{\phi}_0 \in \mathcal{M}_{\mathrm{cl}}$ are the same as the $\mathfrak{h}$-anomalies at the same point.

$\mathrm{Lie}\,(G)\,\phi_0$ is an invariant subspace under $\mathfrak{h}^*$, and the restriction of $\mathfrak{h}^*$ to $\mathrm{Lie}\,(G)\,\phi_0$ is the adjoint action by $\mathfrak{g}\,(\mathfrak{h})$. This can be seen by direct computation. Take any element $\mathfrak{g}\phi_0 \in \mathrm{Lie}\,(G)\,\phi_0$. Then

$$\mathfrak{h}^*\mathfrak{g}\phi_0 = \mathfrak{h}\mathfrak{g}\phi_0 + \mathfrak{g}\,(\mathfrak{h})\,\mathfrak{g}\phi_0 = \mathfrak{g}\mathfrak{h}\phi_0 + \mathfrak{g}\,(\mathfrak{h})\,\mathfrak{g}\phi_0 = [\mathfrak{g}\,(\mathfrak{h})\,,\mathfrak{g}]\phi_0 = Ad_{\mathfrak{g}(\mathfrak{h})}\,\mathfrak{g}\phi_0, \tag{8}$$

since the flavor and gauge symmetries commute, and using Eq. (5). The space U can be broken up into the tangent space to the G-orbit $T_{\phi_0}G\phi_0 \cong \mathrm{Lie}\,(G)$ and its invariant complement, C_{ϕ_0}, since G is reductive. By (b), the map π'_{ϕ_0} is a bijective linear map from C_{ϕ_0} to the tangent space $T_{\hat{\phi}_0}\mathcal{M}_{\rm cl}$ of the moduli space $\mathcal{M}_{\rm cl}$ at $\hat{\phi}_0$, and commutes with $\mathfrak{h}^*$. Thus the action of $\mathfrak{h}^*$ on C_{ϕ_0} is equivalent to the action of $\mathfrak{h}$ on $T_{\hat{\phi}_0}\mathcal{M}_{\rm cl}$, by the similarity transformation S given by π'_{ϕ_0} restricted to C_{ϕ_0}. One can write

$$\mathfrak{h}^* = \mathfrak{h}_{\rm UV} + \mathfrak{g}\,(\mathfrak{h}) = \begin{pmatrix} S\,\mathfrak{h}_{\rm IR}\,S^{-1} & 0 \\ 0 & Ad_{\mathfrak{g}(\mathfrak{h})} \end{pmatrix}, \tag{9}$$

where the second form shows the structure of $\mathfrak{h}^*$ on $U = C_{\phi_0} \oplus T_{\phi_0}G\phi_0$. The action of $\mathfrak{h}$ on U has been labeled by the subscript UV, and the action on the moduli space has been labeled by IR.

One can now compare anomalies in the UV and IR theories using the two different forms for $\mathfrak{h}^*$. Since the adjoint representation is real, the $(\mathfrak{h}^*)^3$ flavor anomaly and $\mathfrak{h}^*$ gravitation anomaly are equal to the anomalies in the infrared theory. All that remains is the proof that the $(\mathfrak{h}^*)^3$ and $\mathfrak{h}^*$ anomalies of U equal the $\mathfrak{h}^3$ and $\mathfrak{h}$ anomalies of U. Let $\mathfrak{h}^{\rm A,B,C}$ be any three elements of $\mathrm{Lie}\,(H)$. Then

$$\begin{aligned}
\mathrm{Tr}\,\mathfrak{h}^{*\rm A}\left\{\mathfrak{h}^{*\rm B},\mathfrak{h}^{*\rm C}\right\} = \; & \mathrm{Tr}\,\mathfrak{h}^{\rm A}_{\rm UV}\left\{\mathfrak{h}^{\rm B}_{\rm UV},\mathfrak{h}^{\rm C}_{\rm UV}\right\} \\
& + \mathrm{Tr}\,\mathfrak{g}\left(\mathfrak{h}^{\rm A}\right)\left\{\mathfrak{h}^{\rm B}_{\rm UV},\mathfrak{h}^{\rm C}_{\rm UV}\right\} + \text{cyclic} \\
& + \mathrm{Tr}\,\mathfrak{h}^{\rm A}_{\rm UV}\left\{\mathfrak{g}\left(\mathfrak{h}^{\rm B}\right),\mathfrak{g}\left(\mathfrak{h}^{\rm C}\right)\right\} + \text{cyclic} \\
& + \mathrm{Tr}\,\mathfrak{g}\left(\mathfrak{h}^{\rm A}\right)\left\{\mathfrak{g}\left(\mathfrak{h}^{\rm B}\right),\mathfrak{g}\left(\mathfrak{h}^{\rm C}\right)\right\}
\end{aligned} \tag{10}$$

The last three lines vanish because the original theory had no gauge and gravitational anomalies, and the flavor symmetries have no gauge anomalies. Thus the $\mathfrak{h}^3$ and $(\mathfrak{h}^*)^3$

anomalies are the same. Similarly the $\mathfrak{h}^*$ and $\mathfrak{h}$ anomalies agree since $\mathfrak{g}$ is traceless because there is no gravitational anomaly. Thus 't Hooft's consistency condition for the flavor anomalies is satisfied.

We now prove the matching theorem for anomalies involving the R-charge using an argument similar to the one presented above. The R-charge acting on U is given by the matrix $\mathfrak{r}$. The R-charge is defined acting on chiral superfields, and so is the charge of the scalar component. Anomalies are computed using the fermionic components, so it is convenient to define a new charge $\tilde{\mathfrak{r}}$ which we will call fermionic R-charge, defined by

$$\tilde{\mathfrak{r}} = \mathfrak{r} - 1. \tag{11}$$

The anomaly can be computed by taking traces over the chiral superfields of $\tilde{\mathfrak{r}}$. The reason for making the distinction between $\mathfrak{r}$ and $\tilde{\mathfrak{r}}$ is that the map π from U to $\mathcal{M}_{\rm cl}$ commutes with $R = \exp \mathfrak{r}$, but does not commute with $\tilde{R} = \exp \tilde{\mathfrak{r}}$.

Assume that R is unbroken at $\hat{\phi}_0 = \pi\left(\phi_0\right)$. Then by an argument similar to that above, it is possible to define a "star" R-charge, $\mathfrak{r}^*$,

$$\mathfrak{r}^* \equiv \mathfrak{r} + \mathfrak{g}\left(\mathfrak{r}\right) \tag{12}$$

which has the form

$$\mathfrak{r}^* = \mathfrak{r}_{\rm UV} + \mathfrak{g}\left(\mathfrak{r}\right) = \begin{pmatrix} S\, \mathfrak{r}_{\rm IR}\, S^{-1} & 0 \\ 0 & Ad_{\mathfrak{g}(\mathfrak{r})} \end{pmatrix} \tag{13}$$

under the decomposition of U into $C_{\phi_0} \oplus T_{\phi_0} G \phi_0$. As in Eq. (9), we have used the subscripts UV and IR to denote the R charges in the ultraviolet and infrared theories. Note that S is the same matrix in Eqs. (9,13), given by π'_{ϕ_0} restricted to C_{ϕ_0}. The fermionic R-charge is then given by

$$\tilde{\mathfrak{r}}^* = \mathfrak{r}^* - 1 = \tilde{\mathfrak{r}}_{\rm UV} + \mathfrak{g}\left(\mathfrak{r}\right) = \begin{pmatrix} S\, \tilde{\mathfrak{r}}_{\rm IR}\, S^{-1} & 0 \\ 0 & Ad_{\mathfrak{g}(\mathfrak{r})} - 1 \end{pmatrix} \tag{14}$$

where in the last equality we have used the fact that fermion R charge $\tilde{\mathfrak{r}}_{\rm IR} = \mathfrak{r}_{\rm IR} - 1$ in the infrared theory.

Compute the trace of $(\tilde{\mathfrak{r}}^*)^3$ in U,

$$\mathrm{Tr}\ (\tilde{\mathfrak{r}}^*)^3 = \mathrm{Tr}\ \{\tilde{\mathfrak{r}}_{\mathrm{UV}} + \mathfrak{g}\,(\mathfrak{r})\}^3 = \mathrm{Tr}\ \left\{\tilde{\mathfrak{r}}_{\mathrm{UV}}^3 + 3\tilde{\mathfrak{r}}_{\mathrm{UV}}^2\mathfrak{g}\,(\mathfrak{r}) + 3\tilde{\mathfrak{r}}_{\mathrm{UV}}\mathfrak{g}\,(\mathfrak{r})^2 + \mathfrak{g}\,(\mathfrak{r})^3\right\}. \tag{15}$$

The R-charge has no gauge anomaly, so $\mathrm{Tr}_U\, \tilde{\mathfrak{r}}_{\mathrm{UV}}\{\mathfrak{g}_{\mathrm{A}}, \mathfrak{g}_{\mathrm{B}}\} + \mathrm{Tr}_{\,\mathrm{Lie}(G)}\, \{Ad_{\mathfrak{g}_{\mathrm{A}}}, Ad_{\mathfrak{g}_{\mathrm{B}}}\} = 0$, for any $\mathfrak{g}_{\mathrm{A,B}} \in \mathrm{Lie}\,(G)$. Here the first term is the matter contribution to the anomaly, and the second term is the gaugino contribution. The absence of gauge anomalies implies that odd powers of $\mathfrak{g}\,(\mathfrak{r})$ vanish when traced over the matter fields, since there is no gaugino contribution to these anomalies. Thus we find

$$\mathrm{Tr}_U\, (\tilde{\mathfrak{r}}^*)^3 = \mathrm{Tr}_U\, (\tilde{\mathfrak{r}}_{\mathrm{UV}})^3 - 3\,\mathrm{Tr}_{\,\mathrm{Lie}(G)}\, Ad^2_{\mathfrak{g}(\mathfrak{r})}. \tag{16}$$

The block diagonal form of $\tilde{\mathfrak{r}}^*$ Eq. (14), gives

$$\mathrm{Tr}_U\, (\tilde{\mathfrak{r}}^*)^3 = \mathrm{Tr}\ (\tilde{\mathfrak{r}}_{\mathrm{IR}})^3 - \mathrm{Tr}_{\,\mathrm{Lie}(G)} \left(1 + 3Ad^2_{\mathfrak{g}(\mathfrak{r})}\right). \tag{17}$$

The R^3 anomaly $A_{\mathrm{UV}}\,(R^3)$ in the UV theory is given by adding the matter and gaugino contributions

$$A_{\mathrm{UV}}\left(R^3\right) = \mathrm{Tr}_U\, (\tilde{\mathfrak{r}}_{\mathrm{UV}})^3 + \mathrm{Tr}_{\,\mathrm{Lie}(G)} 1^3 = \mathrm{Tr}_U\, (\tilde{\mathfrak{r}}^*)^3 + \mathrm{Tr}_{\,\mathrm{Lie}(G)} \left(1 + 3Ad^2_{\mathfrak{g}(\mathfrak{r})}\right). \tag{18}$$

The R^3 anomaly $A_{\mathrm{IR}}\,(R^3)$ in the IR theory is given by

$$A_{\mathrm{IR}}\left(R^3\right) = \mathrm{Tr}\ (\tilde{\mathfrak{r}}_{\mathrm{IR}})^3, \tag{19}$$

since there are no gauginos in the low energy theory. Combining Eq. (16–19), one sees immediately that the UV and IR anomalies are equal, $A_{\mathrm{UV}}\,(R^3) = A_{\mathrm{IR}}\,(R^3)$.

It is straightforward to check that the gravitational R anomaly, and the H^2R and HR^2 anomalies match by a similar computation; the details are given in Ref. [4].

The results derived above allow one to study the matching of anomalies between the ultraviolet and infrared theories at certain points in the classical moduli space. We now derive some results that allow one to relate the anomalies at different points on the moduli space to each other. The moduli space is no longer restricted to be the classical moduli

space $\mathcal{M}_{\rm cl}$. The first case we will consider is when the moduli space $\mathcal{M}$ is an algebraic set in an ambient vector space V given as the critical points of a flavor symmetric superpotential W with R-charge two,

$$\mathcal{M} = \left\{ \hat{\phi} \in V \,|\, W_i(\hat{\phi}) = 0 \right\}, \tag{20}$$

where $\hat{\phi}$ denotes a point in V, and we will use the notation $W_i \equiv \partial W/\partial\hat{\phi}^i$, $W_{ij} \equiv \partial^2 W/\partial\hat{\phi}^i\partial\hat{\phi}^j$, etc. The tangent space to $\mathcal{M}$ at $\hat{\phi}_0$, $T_{\hat{\phi}_0}\mathcal{M}$, is defined by

$$T_{\hat{\phi}_0}\mathcal{M} = \left\{ \hat{v}^i \in V \,|\, W_{ij}(\hat{\phi}_0)\hat{v}^j = 0 \right\}. \tag{21}$$

In all the cases we are interested in, W is a polynomial in $\hat{\phi}$ and Eq. (21) agrees with the algebraic geometry notion of the tangent space.

Assume that a subgroup H (not containing an R symmetry) of the flavor symmetry group F is unbroken at a point $\hat{\phi}_0 \in \mathcal{M}$. The invariance of the superpotential W under F implies that

$$W\left(h^i_j\,\hat{\phi}^j\right) = W(\hat{\phi}^i), \tag{22}$$

where h^i_j is the matrix for the H transformation in the representation ρ of the fields $\hat{\phi}$. Differentiating this equation twice with respect to $\hat{\phi}$ and using $H\hat{\phi}_0 = \hat{\phi}_0$ gives

$$h^k_i\, h^l_j\, W_{kl}\left(\hat{\phi}_0\right) = W_{ij}(\hat{\phi}_0), \tag{23}$$

which shows that $W_{ij}(\hat{\phi}_0)$ is a H invariant tensor that transforms as $(\bar{\rho}\otimes\bar{\rho})_S$ under H. The tangent space to $\mathcal{M}$ at $\hat{\phi}_0$ is the null-space of W_{ij}, and so is H-invariant. One can write $V = T_{\hat{\phi}_0}\mathcal{M} + N_{\hat{\phi}_0}\mathcal{M}$ as the direct sum of the tangent space and its orthogonal complement in V. Then W_{ij} provides a non-singular invertible map from $N_{\hat{\phi}_0}\mathcal{M}$ into its dual, so that $N_{\hat{\phi}_0}\mathcal{M}$ transforms as a real representation of H. This immediately implies that the H anomalies computed using the flat directions $T_{\hat{\phi}_0}\mathcal{M}$ agree with those computed using the entire vector space V.

A similar result holds for the anomalies involving the R charge. Let R_i be the R-charge of $\hat{\phi}_i$, so that

$$W\left(e^{i\alpha R_i}\,\hat{\phi}^i\right) = e^{2i\alpha}\,W(\hat{\phi}^i), \tag{24}$$

since W has R charge two. Differentiating twice with respect to $\hat{\phi}$ shows that

$$e^{i\alpha(R_i+R_j)}\,W_{ij}(\hat{\phi}_0) = e^{2i\alpha}\,W_{ij}(\hat{\phi}_0), \tag{25}$$

which can be written in the suggestive form

$$e^{i\alpha([R_i-1]+[R_j-1])}\,W_{ij}(\hat{\phi}_0) = W_{ij}(\hat{\phi}_0). \tag{26}$$

R_i-1 is the R charge of the fermionic component of the chiral superfield. Thus Eq. (26) shows that $N_{\hat{\phi}_0}\mathcal{M}$ transforms like a real representation under $\tilde{R} = R-1$, the fermionic R charge. Thus the R anomalies, (and mixed anomalies involving R and non-R flavor symmetries) can be computed at $\hat{\phi}_0$ using V instead of $T_{\hat{\phi}_0}\mathcal{M}$. The result can be summarized by

Theorem III: Let $\mathcal{M} \subseteq V$ be a moduli space described by the critical points of a flavor symmetric superpotential W with R-charge two. Then the anomalies of an unbroken subgroup $H \subseteq F$ at a point $\hat{\phi}_0 \in \mathcal{M}$ can be computed using the entire space V, instead of $T_{\hat{\phi}_0}\mathcal{M}$. If the anomaly matching conditions between the UV and IR theories for H are satisfied at $\hat{\phi}_0$, they are also satisfied at all points of any moduli space $\mathcal{M}' \in V$ given by the critical points of any W' (including $W' = 0$ and $W' = W$).

Note that this result tells us that for moduli spaces described by invariant superpotentials, the precise form of the moduli space is irrelevant. The only role of possible quantum deformations is to remove points of higher symmetry from the moduli space. It also greatly simplifies the computation of anomalies in the infrared theory, since one does not need to compute the tangent vectors at a given point in the moduli space.

One simple application of the above result is to prove that anomaly matching conditions are compatible with integrating out heavy fields. Assume that one has a theory with a moduli space $\mathcal{M}_\Lambda$ described by a superpotential $W(\hat{\phi}, \Lambda)$. Now perturb the UV theory by adding a tree level mass term $m_{ij}\phi^i\phi^j$ to the superpotential. $m_{ij}\phi^i\phi^j$ is gauge invariant, and can be written as a polynomial $W_m(\hat{\phi})$ of the gauge invariant composites $\hat{\phi}$ of the IR theory.

If the UV theory contains no singlets, then $W_m(\hat{\phi})$ is linear in the basic gauge invariant composite fields $\hat{\phi}$. From this, it immediately follows that the effective superpotential of the massive theory is given by

$$W(\hat{\phi}, \Lambda) = W_0(\hat{\phi}, \Lambda) + W_m(\hat{\phi}), \tag{27}$$

where W_0 is the superpotential in the absence of a mass term, since a linear term in the fields is equivalent to a redefinition of the source.

The anomalies in the IR theory for any unbroken subgroup are unaffected by the change in the moduli space due to the addition of the mass term. They are still obtained by tracing over the whole space V. In the UV theory, one should trace not over the whole space U, but only over the modes that remain massless when W_m is turned on. But it is easy to see that the massive modes in the UV theory form a real representation of the unbroken symmetry. The argument is the same as that used in the IR theory, except that $W_{ij}(\hat{\phi}_0)$ is replaced by the (constant) matrix m_{ij}. The mass term does not introduce any modifications to the anomaly in the UV or IR theory for any symmetry left unbroken by the mass. Thus one finds that if the 't Hooft conditions are verified for a theory with a moduli space given by a superpotential, they are also valid for any theory obtained by integrating out fields by adding a mass term.

One can now apply the results to study anomaly matching in supersymmetric gauge theories. Consider supersymmetric QCD with $N_F \geq N_c > 2$. The fundamental fields are the quarks $Q^{i\alpha}$ and antiquarks $\tilde{Q}_{j\beta}$. The flavor symmetry group is $SU(N_F)_L \times SU(N_F)_R \times U(1)_B \times U(1)_R$ if $N_c > 2$.[1]

Consider the point ϕ_0 in the UV theory

$$Q^{i\alpha} = \begin{cases} m\delta^{i\alpha} & i \leq N_c \\ 0 & i > N_c \end{cases}, \qquad \tilde{Q}_{j\alpha} = 0. \tag{28}$$

[1]For $N_c = 2$, the $SU(N_F)_L \times SU(N_F)_R \times U(1)_B$ is enlarged into a $SU(2N_F)$ flavor symmetry. Anomaly matching can be proven by an argument similar to that for $N_c > 2$.

The point $\pi(\phi_0) = \hat{\phi}_0$ in the IR theory is described by gauge invariant meson and baryons fields,

$$M^i_j = 0, \qquad \tilde{B}^{j_1 \cdots j_s} = 0, \qquad B_{i_1 \cdots i_s} = m^{N_c}\ \epsilon_{12 \ldots N_c i_1 \cdots i_s}. \tag{29}$$

The unbroken flavor group at $\hat{\phi}_0$ is $SU(N_c)_L \times SU(N_F - N_c)_L \times SU(N_F)_R \times U(1)_B \times U(1)_R$. Under these unbroken symmetries, the fields transform as

	$SU(N_c)_L$	$SU(N_F - N_c)_L$	$SU(N_F)_R$	$U(1)_B$	$U(1)_R$
$Q^{i\alpha}$, $i \le N_c$	N_c	$-$	$-$	0	0
$Q^{i\alpha}$, $i > N_c$	$-$	$N_F - N_c$	$-$	$-N_F$	$(3N_F - 4N_c)/(2N_F - N_c)$
$\tilde{Q}_{j\alpha}$	$-$	$-$	$\overline{N}_F$	$N_F - N_c$	$(3N_F - 4N_c)/(2N_F - N_c)$

The unbroken $U(1)_B$ and $U(1)_R$ symmetries are linear combinations of the original $U(1)_B$ and $U(1)_R$ and a $U(1)$ generator in $SU(N_F)_L$.

The point $\phi_0 = (Q^{i\alpha}, Q_{j\alpha})$ breaks the gauge group completely. The orbit $G\phi_0$ is closed and has maximal dimension, so Theorem I tells us that the hypotheses of Theorem II are satisfied. The anomaly matching theorem (Theorem II) implies that the $SU(N_c)_L \times SU(N_F - N_c)_L \times SU(N_F)_R \times U(1)_B \times U(1)_R$ anomalies must match between the UV and IR theories. It is straightforward to verify by explicit computation that this is the case. The UV anomalies are computed using the above transformation rules for the fundamental fields. The IR anomalies are computed by determining the representation of the tangent vectors to the classical moduli space at $\hat{\phi}_0$ under the unbroken symmetry. One can similarly show that anomaly matching holds at other points at the moduli space of supersymmetric QCD.

In the special case $N_F = N_c + 1$, the classical moduli space is described by a superpotential. Then Theorem III implies that since the $SU(N_c)_L \times SU(N_F - N_c)_L \times SU(N_F)_R \times U(1)_B \times U(1)_R$ match at Eq. (29), they must also match at the origin. One can verify by explicit computation that these anomalies match at the origin for $N_F = N_c + 1$, but not for any other value of N_F. (Equivalently, the fact that the these anomalies match at Eq. (29) but not at the origin for $N_F > N_c + 1$ implies that the moduli space for $N_F > N_c + 1$ cannot be given by a superpotential.) Similarly by considering the point given by exchanging the values of Q and

$\tilde{Q}$ in Eq. (29), one can prove that the $SU(N_F)_L \times SU(N_c)_R \times SU(N_F - N_c)_R \times U(1)_B \times U(1)_R$ anomalies also match at the origin. Anomaly matching for these two subgroups is sufficient to guarantee that the anomalies for the full $SU(N_F)_L \times SU(N_F)_R \times U(1)_B \times U(1)_R$ flavor group match at the origin. Applying Theorem III again then shows that the anomalies match everywhere on the moduli space.

Since supersymmetric QCD with $N_F = N_c + 1$ is described by a superpotential, integrating out one flavor by adding a mass term also gives a consistent theory. This is supersymmetric QCD with $N_F = N_c$ with the quantum deformed moduli space $\det M - B\tilde{B} = \Lambda^{2N_c}$.[2] Integrating out additional flavors leads to a trivial result, since there is no point in the moduli space of the theory, and all vacua are unstable because of the quantum superpotential [6]. One cannot relate anomalies in $N_F > N_c + 1$ to those for $N_F = N_c + 1$ by adding mass terms, since the theories with $N_F > N_c + 1$ are not described by a superpotential. This is consistent with the result that these theories do not satisfy the anomaly matching conditions at the origin, and the infrared behavior is governed by a dual theory [1].

The results of this paper have been used to reproduce known results for supersymmetric QCD, without having to explicitly compute any anomalies in the UV or IR theories. The key point is to find some simple field configurations ϕ that completely break the gauge symmetry, and at which π' is surjective. The results are particularly powerful for theories with a moduli space described by a superpotential. The results of this paper can also be applied to the s-confining theories that have been studied recently [3]. These theories have a moduli space given by a superpotential, and are therefore similar to supersymmetric QCD for $N_F = N_c + 1$. The 't Hooft consistency conditions are automatically satisfied for the entire moduli space, using Theorems I–III. It then follows that any theory obtained from an s-confining theory by adding mass terms also satisfies the 't Hooft consistency conditions,

[2]Note that the anomaly matching theorem cannot be applied at the origin, which is a point of the classical moduli space, but is not part of the quantum moduli space.

as long as it has supersymmetric vacua. A direct check of the anomaly matching conditions by explicit computation is extremely involved.

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